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LETTER TO THE EDITOR

Plethysm problem of U((n+1)(n+2)/2)⊃SU(3)

V K B Kota
Physical Research Laboratory, Navrangpura, Ahmedabad 380009, India

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Abstract. A method of reducing the partitions of the unitary groups U((n+1)(n+2)/2) to irreducible SU(3) contents is presented.

1. Introduction

The reduction of a unitary group U(*N*) to its subgroups is a well defined group-theoretical problem. Apart from its mathematical significance, it has profound application to physical problems since the spectroscopic space generated by a single particle in a central potential allows one to introduce a wide variety of unitary symmetries. The class of unitary groups U(*S*) with $S = (n+1)(n+2)/2$, *n* being an integer, is of particular interest as these groups are the orbital symmetry groups of a three-dimensional oscillator (Moshinsky 1969). The reduction of these groups (to be denoted by U(*S*)) to their irreducible SU(3) contents is of fundamental importance to nuclear physics problems in particular (Elliott 1958, Ratna Raju *et al* 1972) and to other branches of physics in general (Wybourne 1970, Haskell *et al* 1971). The purpose of this letter is to present a method of obtaining these reductions.

2. Method of calculation

The reduction of the partitions of U(*S*) to irreducible representations (IR) of SU(3) could be accomplished in a straightforward way using the concept of a plethysm of *S*-functions introduced by Littlewood (1936). Here one makes use of the expression (Todd 1949),

$$\{A\}\otimes\{f\}=\frac{1}{r!}\sum_k h_k C_k^{[f]}(\{A\}\otimes S_1)^{\alpha}(\{A\}\otimes S_2)^{\beta}\ldots . \qquad (1)$$

In the above expression *k* is the class of the symmetric group of order *r*! corresponding to the partition [*f*] of the integer *r* specified by the cyclic structure

$$S_k = S_1^{\alpha} S_2^{\beta}\ldots$$

h_k is the order of the class *k*, $C_k^{[f]}$ is the character of the class *k* corresponding to the partition [*f*].

The dimensionality of the partition [*n* 0 0] of U(3) is $(n+1)(n+2)/2$. Thus, substituting in equation (1) the *S*-function $\{n\}$ in place of the *S*-function $\{A\}$ one obtains the reductions of the partitions [*f*] of U(*S*).

Application of equation (1), though straightforward, is very difficult for large values of the integer r as S-function outer-multiplication becomes tremendously laborious. This is the reason why only reductions of partitions of $r \leqslant 6$ of U(10) (Ibrahim 1950) and $r \leqslant 8$ of U(15) (Krishna Brahmam and Ratna Raju 1975) appear in the literature. Also the non-availability of characters beyond $r > 16$ puts a restriction on the above method. Therefore, a search is made for a simplified procedure which we will discuss below.

2.1. *Method of obtaining* IR *of* SU(3) *in the product of two S-functions*

The possible IR of U(3) for a given integer r of U(S) are given by the partitions $[\lambda_1 \lambda_2 \lambda_3]$. Here λ_1, λ_2 and λ_3 are any integers satisfying the conditions

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0 \tag{2}$$

and

$$\lambda_1 + \lambda_2 + \lambda_3 = n*r \tag{3}$$

where n is related to S by the relation

$$S = (n+1)(n+2)/2.$$

The corresponding SU(3) representations $(\lambda\ \mu)$ are given by

$$(\lambda\ \mu) = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3). \tag{4}$$

For $r < 3$, besides the conditions (2)–(4), one will have further restrictions that

$$\lambda_3 = 0, \qquad \text{for } r = 2 \tag{5}$$

and

$$\lambda_2 = \lambda_3 = 0, \qquad \text{for } r = 1. \tag{6}$$

The problem of reducing a partition $[X]$ of the integer x of U(S) is just the problem of finding the number of times each $(\lambda\ \mu)$, given by equations (2)–(6), is contained in the given partition. This we will write in a mathematical notation as

$$\{n\} \otimes \{X\} = \sum_{r'} A_{xr'}(\lambda_{xr'}\ \mu_{xr'}) \tag{7}$$

where $(\lambda_{xr'}\ \mu_{xr'})$ are all possible IR of SU(3) for the integer x of U(S) and $A_{xr'}$ is the number of times the SU(3) representation $(\lambda_{xr'}\ \mu_{xr'})$ occurs in the reduction of the partition $[X]$ of U(S).

A similar expression for a partition $[Y]$ of the integer y can be written as

$$\{n\} \otimes \{Y\} = \sum_{r''} A_{yr''}(\lambda_{yr''} \mu_{yr''}). \tag{8}$$

Now we will see how to find IR of SU(3) contained in the product of the partitions $[X]$ and $[Y]$. This problem is just that of finding $A_{zr'''}$ in the expression

$$\{n\} \otimes [\{X\}\{Y\}] = \sum_{r'''} A_{zr'''}(\lambda_{zr'''}\ \mu_{zr'''}). \tag{9}$$

It is obvious that $z = x + y$.

To find $A_{zr'''}$ we first find the number of times the SU(3) representation $(\lambda_{zr'''}\ \mu_{zr'''})$ occurs in the Kronecker product of the SU(3) representations $(\lambda_{xr'}\ \mu_{xr'}) \times (\lambda_{yr''}\ \mu_{yr''})$ and multiply it with $A_{xr'} * A_{yr''}$. Then we repeat the same for all r' and then for all r''. If we add them all we get $A_{zr'''}$. If we repeat the above procedure for all r''', all $A_{zr'''}$ will be obtained. Here one needs to know how to find the number of times a given SU(3) representation $(\lambda_3\ \mu_3)$ occurs in the Kronecker product of any two arbitrary SU(3) representations $(\lambda_1\ \mu_1)\times(\lambda_2\ \mu_2)$. This can be accomplished by using the prescription given by Chew and Sharp (1966).

2.2 *Method of obtaining* IR *of* SU(3) *in a given partition of* U(S)

Knowing how to find IR of SU(3) in the product of any two S-functions of U(S), one can get the IR of SU(3) in a given partition of U(S) as follows.

Expand the partition to products of the partitions of the type $[1^r]$ making use of the determinant expansion of S-functions (Littlewood 1940). Now the products can be reduced to IR of SU(3) by using the method explained in § 2.1.

However, to apply the above method one needs the reductions of all possible partitions of the type $[1^r]$ of U(S). Actually only half of them need be reduced, as the other half can be obtained using the relation

$$[1^{S-r}]=\sum_r A_r(\lambda_r\mu_r) \qquad \text{if} \qquad [1^r]=\sum_r A_r(\mu_r\lambda_r). \tag{10}$$

The partitions $[1^r]$ can be reduced easily by using a simple counting procedure suggested by K T Hecht (1974, private communication). Alternatively one can apply the method described in § 2.1. to equation (1). Here it is trivial to write down the characters $C_k^{[f]}$. And one can use the reciprocity theorem (Weyl 1930) to obtain an expression for $\{n\}\otimes S_r$ as

$$\begin{aligned}\{n\}\otimes S_r=\sum_{a,b}[&\{nr-ar, ar-br, br\}-\{nr-ar, ar-br-1, br+1\}\\ &+\{nr-ar-1, ar-br-1, br+2\}-\{nr-ar-1, ar-br+1, br\}\\ &+\{nr-ar-2, ar-br+1, br+1\}-\{nr-ar-2, ar-br, br+2\}].\end{aligned} \tag{11}$$

In the above expression, the summation is over all positive integers a and b with the constraint that all non-standard S-functions are to be completely ignored. The S-functions $\{\lambda_1\,\lambda_2\,\lambda_3\}$ where λ_1, λ_2 and λ_3 violate the condition (2) for $r>3$, the conditions (2) and (5) for $r=2$ and the conditions (2) and (6) for $r=1$, are all non-standard S-functions. It is important to note that equation (11) gives only the U(3) content of $\{n\}\otimes S_r$ as needed for applications to SU(3).

3. Conclusions

The method of obtaining IR of SU(3) in a partition of U(S) as described in this letter can be translated to a machine code in a straightforward way. The programs have been developed and they may be published elsewhere.

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